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L^2-rigidity in von Neumann algebras

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Abstract. We introduce the notion of L^2-rigidity for von Neumann algebras, a generalization of property (T) which can be viewed as an analogue for the vanishing of 1-cohomology into the left regular representation of a group. We show that L^2-rigidity passes to normalizers and is satisfied by nonamenable II_1 factors which are non-prime, have property Γ, or are weakly rigid. As a consequence we obtain that if M is a free product of diffuse von Neumann algebras, or if $M = L\Gamma$ where Γ is a finitely generated group with $\beta_1^{(2)}(\Gamma) > 0$, then any nonamenable regular subfactor of M is prime and does not have properties Γ or (T). In particular this gives a new approach for showing solidity for a free group factor thus recovering a well known recent result of N. Ozawa.

1. Introduction

In their pioneering work of the 80's Connes and Jones ([C3], [CJ]) introduced the notion of property (T) (or rigidity) for II_1 factors by requiring that any sequence of subunital, subtracial completely positive maps which converge pointwise in $\|\cdot\|_2$ to the identity also converge uniformly in $\|\cdot\|_2$ to the identity on $(N)_1$. This type of rigidity phenomenon (and it's relative version later introduced by Popa [P4]) has since led to the solution of many old problems in von Neumann algebras and orbit equivalence ergodic theory ([C2], [IPP], [P3], [P4], [P5]). In [Pe] it was shown that property (T) is equivalent to a vanishing 1-cohomology type result for closable derivations into arbitrary Hilbert bimodules. This equivalence is achieved in part by using Sauvageot's results ([S1], [S2], [CiS]) which state that there is a bijective correspondence between densely defined real closable derivations into Hilbert bimodules and semigroups of unital, tracial completely positive maps.

For an inclusion of finite von Neumann algebras ($N \subset M$) one cannot hope to obtain such a cohomological characterization of relative property (T) (even if N itself has property (T)) as there is no guarantee that a closed derivation δ on M is even densely defined on N much less inner. However we will always have that the associated semigroup will converge uniformly in $\|\cdot\|_2$ to id on $(N)_1$ and thus we may interpret this fact as saying that δ "vanishes" on N.

In this paper we will use the above techniques to investigate closable derivations into the coarse correspondence $L^2(N)\ \overline{\otimes}\ L^2(N)$. We will say that an inclusion of finite von Neumann algebra ($B \subset N$) is L^2-rigid if all derivations which arise in this way "vanish" in the above sense on B, and we will say that a finite von Neumann algebra N is L^2-rigid if the inclusion ($N \subset N$) is L^2-rigid, (see Definition 4.1 for the precise definition). Derivations into the coarse correspondence appear naturally in the context of Voiculescu's nonmicrostates approach to free entropy [V], and also play a central role in studying the first L^2-Betti number of a von Neumann algebra as introduced by Connes and Shlyakhtenko [CSh] (see also [T]). This should be compared to the situation for groups where Bekka and Valette [BV] have shown that for a finitely generated nonamenable group the first L^2-Betti number vanishes if and only if the first cohomology group into the left regular representation vanishes.

We will show that given a nonamenable subfactor $Q \subset N$ and a densely defined real closable derivation from N into $(L^2(N)\ \overline{\otimes}\ L^2(N))^{\oplus\infty}$ then the derivation must "vanish" on $Q' \cap N$. Furthermore we will show that from the mixing property of the coarse correspondence that if $Q' \cap N$ is diffuse then we further have that the derivation "vanishes" on $W^*(\mathcal{N}_N(Q' \cap N))$. Taking a free ultrafilter ω, and using a slight modification of the above arguments using N^ω we will also show that if N is a nonamenable factor which has property Γ of Murray and von Neumann [MvN] then any derivation as above must "vanish" on N. Recall that a II_1 factor N is non-prime if $N = Q\ \overline{\otimes}\ B$ where both Q and B are infinite dimensional. The main result is the following:

Theorem 1.1 *Let N be a II_1 factor which is non-prime or has property Γ, then N is L^2-rigid.*

The above theorem shows that L^2-rigidity is a very weak rigidity type phenomenon (for instance $R\ \overline{\otimes}\ L\mathbb{F}_2$ is L^2-rigid even though it has Haagerup's compact approximation property [H1]). On the other hand we will see that if N is a free product of diffuse finite von Neumann algebras or if $N = L\Gamma$ where Γ is a finitely generated group with $\beta_1^{(2)}(\Gamma) > 0$, then N is not L^2-rigid.

In [P1] Popa showed that for the uncountable free groups, their group factors are prime. Using techniques from Voiculescu's free probability this was shown by Ge to also be the case for countable free groups [Ge]. This was generalized to all free products of diffuse finite von Neumann algebras which embed into R^ω by Jung [J].

Recall that if M is a finite von Neumann algebra, then a subalgebra $B \subset M$ is said to be regular in M if the set of unitaries $u \in \mathcal{U}(M)$ which satisfy $uBu^* = B$ generates M as a von Neumann algebra. From the above remarks we have the following:

Theorem 1.2 *Let M be a free product of diffuse finite von Neumann algebras or $M = L\Gamma$ where Γ is a finitely generated group with $\beta_1^{(2)}(\Gamma) > 0$, then any regular nonamenable subfactor of M is prime and does not have properties Γ or (T).*

Using techniques from C^*-algebra theory Ozawa was able to show not just that the free group factors are prime but that in fact they are solid [O1], i.e. the commutant of any diffuse subalgebra is amenable. As an application of Theorem 1.1 we obtain a new approach to Ozawa's result using the fact that the free groups have the "L^2-Haagerup property", i.e. there exist proper cocycles into direct sums of the left regular representation.

Theorem 1.3 *Let Γ be a countable discrete group such that there exists a proper cocycle $b : \Gamma \to (\ell^2\Gamma)^{\oplus\infty}$, (for example $\Gamma = \mathbb{F}_n$, $2 \leq n \leq \infty$). Then $L\Gamma$ is solid.*

It should be noted that although the above result gives a new proof of Ozawa's theorem for the case of the free groups, it is a quite different approach than in [O1]. Indeed, we use the fact that Γ has Haagerup's property in a crucial way. Whereas in [O1] the above is shown for all hyperbolic groups, many of which have property (T). We also note that S. Popa in [P6] has recently given another proof of Ozawa's theorem for specific case of the free group factors.

In Sect. 5 we investigate derivations which naturally appear in free products of von Neumann algebras. These derivations give rise to deformations by free products of multiples of the identity, thus we may extend the Kurosh type theorem in [IPP, Theorem 0.1] to include many von Neumann subalgebras which do not have relative property (T). The first Kurosh type theorem in von Neumann algebras was obtained by Ozawa [O2] using C^*-algebra theory.

Theorem 1.4 *Let M_1 and M_2 be finite factors and let $M = M_1 * M_2$. If $Q \subset M$ is a subfactor such that $Q' \cap M$ is a nonamenable factor, or if $Q \subset M$ is a nonamenable subfactor with property Γ and $Q' \cap M$ is a factor, then there exists $i \in \{1, 2\}$ and a unitary operator $u \in \mathcal{U}(M)$ such that $uQu^* \subset M_i$.*

In Sect. 6 we consider the case of a tensor product of II_1 factors $M = M_1 \overline{\otimes} \cdots \overline{\otimes} M_n$, such that each M_i has a derivation into it's coarse correspondence which does not "vanish". We show that if Q is a regular nonamenable subfactor then there exists a corner of $Q' \cap M$ which embeds into M_i' for some $i \leq n$, where M_i' is the von Neumann subalgebra obtained by replacing M_i with $\mathbb{C}$ in the above tensor product. Ozawa and Popa [OP] gave examples of tensor products of von Neumann algebras which have unique prime factorization. Using the conjugacy results in [OP] we are able to give new examples of this type.

Theorem 1.5 *Let M_i be nonamenable II_1 factors $1 \leq i \leq m$, such that each M_i is a non-trivial free product or $L\Gamma$ for some finitely generated group Γ with $\beta_1^{(2)}(\Gamma) > 0$, assume $N_1 \overline{\otimes} \cdots \overline{\otimes} N_n = M_1 \overline{\otimes} \cdots \overline{\otimes} M_m$, for some prime II_1 factors $N_1, \dots, N_n$, then $n = m$ and there exist $t_1, t_2, \dots, t_m > 0$ with $t_1 t_2 \cdots t_m = 1$ such that after permutation of indices and unitary conjugacy we have $N_k^{t_k} = M_k$, $\forall k \leq m$.*

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2. Preliminaries and notation

Suppose N is a finite von Neumann algebra with normal faithful trace τ, $D(\delta) \subset N$ is a weakly dense $*$-subalgebra, $\mathcal{H}$ is an N–N Hilbert bimodule, and $\delta : D(\delta) \to \mathcal{H}$ is a derivation ($\delta(xy) = x\delta(y) + \delta(x)y$, $\forall x, y \in D(\delta)$), which is closable (as an unbounded operator from $L^2(N, \tau)$ to $\mathcal{H}$), and real ($\langle \delta(x), y\delta(z) \rangle = \langle \delta(z^*)y^*, \delta(x^*) \rangle$, $\forall x, y, z \in D(\delta)$).

It follows from [S1] and [DL] that $D(\overline{\delta}) \cap N$ is a $*$-subalgebra of N and $\overline{\delta}|_{D(\overline{\delta}) \cap N}$ is again a derivation. Let $\Delta = \delta^* \overline{\delta}$, then Δ is the generator of a completely Dirichlet form [S1]. By a deformation on a von Neumann algebra we mean a net of completely positive maps which converge to the identity pointwise in $\| \cdot \|_2$. Associated to Δ are two natural deformations of N, the first is the completely positive semigroup (completely Markovian semigroup) $\{\phi_t\}_{t>0}$, each $\phi_t = \exp(-t\Delta)$ is a c.p. map which is unital ($\phi_t(1) = 1$), tracial ($\tau \circ \phi_t = \tau$), and positive ($\tau(\phi_t(x)x^*) \geq 0$, $\forall x \in N$), moreover the semigroup property is satisfied ($\phi_{t+s} = \phi_t \circ \phi_s$, $\forall s, t > 0$), and $\forall x \in N$, $\|x - \phi_t(x)\|_2 \to 0$, as $t \to 0$. The second deformation associated to Δ is the deformation coming from resolvent maps $\{\eta_\alpha\}_{\alpha>0}$, again each $\eta_\alpha = \alpha(\alpha + \Delta)^{-1}$ is a unital, tracial, positive, c.p. map such that $\forall x \in N$, $\|x - \eta_\alpha(x)\|_2 \to 0$, as $\alpha \to \infty$, furthermore $\beta\eta_\alpha - \alpha\eta_\beta = (\beta - \alpha)\eta_\alpha \circ \eta_\beta$, $\forall \alpha, \beta > 0$.

The relationship between these maps are as follows and can be found for example in [MR]:

$$\Delta = \lim_{t \to 0} \frac{1}{t}(\mathrm{id} - \phi_t) = \alpha\big(\eta_\alpha^{-1} - \mathrm{id}\big) = \lim_{\alpha \to \infty} \alpha(\mathrm{id} - \eta_\alpha),$$

$$\phi_t = \exp(-t\Delta) = \lim_{\alpha \to \infty} \exp(-t\alpha(\mathrm{id} - \eta_\alpha)),$$

$$\eta_\alpha = \alpha(\alpha + \Delta)^{-1} = \alpha \int_0^\infty e^{-\alpha t} \phi_t dt.$$

Note that we will use the same symbols Δ, ϕ_t, and η_α for the maps on N as well as the corresponding extensions to $L^2(N, \tau)$. Also note that η_α maps into the domain of Δ and $\Delta \circ \eta_\alpha = \alpha(\mathrm{id} - \eta_\alpha)$. Furthermore we have that $\mathrm{Range}(\eta_\alpha) = D(\Delta) \subset D(\overline{\delta})$, $D(\Delta^{\frac{1}{2}}) = D(\overline{\delta}) = \mathrm{Range}(\eta_\alpha^{1/2})$ and $\forall x \in D(\overline{\delta})$, $\|\Delta^{\frac{1}{2}}(x)\|_2 = \|\delta(x)\|_2$.

If $B \subset N$ is a von Neumann subalgebra we will say that a deformation $\{\Phi_\iota\}_\iota$ converges uniformly on $(B)_1$ if $\forall \varepsilon > 0, \exists \iota_0$ such that $\forall \iota > \iota_0, b \in (B)_1$ we have that $\|b - \Phi_\iota(b)\|_2 < \varepsilon$.

Lemma 2.1 *Let (N, τ) be a finite von Neumann algebra, $B \subset N$ a von Neumann subalgebra, and $\{\phi_t\}_t$, $\{\eta_\alpha\}_\alpha$ deformations as above. The deformation $\{\eta_\alpha\}_\alpha$ converges uniformly on $(B)_1$ as $\alpha \to \infty$ if and only if the deformation $\{\phi_t\}_t$ converges uniformly on $(B)_1$ as $t \to 0$.*

Proof. Since $0 \leq \phi_t \leq \mathrm{id}$, $\forall t > 0$ we have that $\forall x \in N$, $t \mapsto \tau((x - \phi_t(x))x^*)$ is a non-negative valued function, also since

$$\begin{aligned}\tau\big((x - \phi_{t+s}(x))x^*\big) &= \tau\big((x - \phi_t(x))x^*\big)\\ &\quad + \tau\big((\phi_{t/2}(x) - \phi_s(\phi_{t/2}(x)))\phi_{t/2}(x)^*\big)\\ &\geq \tau\big((x - \phi_t(x))x^*\big),\end{aligned}$$

we have that $t \mapsto \tau((x - \phi_t(x))x^*)$ decreases to 0 as $t \to 0$. Hence if $\{\phi_t\}_t$ does not converge uniformly on $(B)_1$ as $t \to 0$ then $\exists c_0 > 0$ such that $\forall t > 0$, $\exists x_t \in (B)_1$, such that $\tau((x_t - \phi_t(x_t))x_t^*) \geq c_0$. Therefore $\tau((x_t - \eta_{1/t}(x_t))x_t^*) = \int_0^\infty e^{-s}\tau((x_t - \phi_{st}(x_t))x_t^*)ds \geq \int_1^\infty e^{-s}c_0 ds = c_0 e^{-1}$, thus $\{\eta_\alpha\}_\alpha$ does not converge uniformly on $(B)_1$ as $\alpha \to \infty$.

Conversely if $\{\phi_t\}_t$ does converge uniformly on $(B)_1$ as $t \to 0$, then $\forall x \in (B)_1$ we have $\|x - \eta_\alpha(x)\|_2 \leq \int_0^\infty e^s \|x - \phi_{s/\alpha}(x)\|_2 ds$ and since $\|x - \phi_t(x)\|_2 \leq 2$, $\forall x \in (B)_1$, $t > 0$ it follows that $\{\eta_\alpha\}_\alpha$ also converges uniformly on $(B)_1$ as $\alpha \to \infty$.

Finally we mention that $\Delta^{\frac{1}{2}}$ also generates a completely Dirichlet form as is shown in [S3] by the formula: $\Delta^{\frac{1}{2}} = \pi^{-1}\int_0^\infty t^{-1/2}(\mathrm{id} - \eta_t)dt$.

Example 2.2 Suppose Γ is a countable discrete group, $\pi : \Gamma \to \mathcal{O}(\mathcal{K})$ is an orthogonal representation, and $b : \Gamma \to \mathcal{K}$ is a 1-cocycle. Then associated to this cocycle is a conditionally negative definite function ψ given by $\psi(\gamma) = \|b(\gamma)\|_2^2$, there is also a semigroup of positive definite functions $\{\varphi_t\}_t$ given by $\varphi_t(\gamma) = e^{-t\psi(\gamma)}$, and the set of positive definite resolvents $\{\chi_\alpha\}_\alpha$ given by $\chi_\alpha(\gamma) = \alpha/(\alpha + \psi(\gamma))$.

Let $\mathcal{H} = \mathcal{K} \overline{\otimes}_{\mathbb{R}} L^2(L\Gamma)$ and equip $\mathcal{H}$ with the $L\Gamma$ bimodule structure which satisfies $u_\gamma(\xi \otimes \xi') = \pi(\gamma)\xi \otimes u_\gamma\xi'$ and $(\xi \otimes \xi')u_\gamma = \xi \otimes \xi' u_\gamma$, $\forall \gamma \in \Gamma, \xi \in \mathcal{H}, \xi' \in L^2(L\Gamma)$. Let $\delta_b : \mathbb{C}\Gamma \to \mathcal{H}$ be the derivation which satisfies $\delta_b(u_\gamma) = b(\gamma) \otimes u_\gamma$, $\forall \gamma \in \Gamma$, then δ_b is a real closable derivation and so as described above we can associate with δ_b the c.c.n. map Δ along with the deformations $\{\phi_t\}_t$ and $\{\eta_\alpha\}_\alpha$. It can be easily checked that we have the following relationships:

$$\Delta(u_\gamma) = \psi(\gamma)u_\gamma, \quad \forall \gamma \in \Gamma,$$
$$\phi_t(u_\gamma) = \varphi_t(\gamma)u_\gamma, \quad \forall \gamma \in \Gamma, t > 0,$$
$$\eta_\alpha(u_\gamma) = \chi_\alpha(\gamma)u_\gamma, \quad \forall \gamma \in \Gamma, \alpha > 0.$$

Note that in this case we have that if $\Lambda < \Gamma$ then the derivation $\delta_{b|\mathbb{C}\Lambda}$ is inner if and only if the cocycle $b_{|\Lambda}$ is inner if and only if the deformation $\{\eta_\alpha\}_\alpha$ converges uniformly on $(L\Lambda)_1$. Note also that if $\mathcal{K}$ is the left regular representation of Γ then $\mathcal{H}$ is the coarse correspondence for $L\Gamma$.

Example 2.3 Suppose (M_1, τ_1) and (M_2, τ_2) are finite diffuse von Neumann algebras, and let $(M, \tau) = (M_1 * M_2, \tau_1 * \tau_2)$. If we let $\delta_i : M_1 *_{\mathrm{Alg}} M_2 \to L^2(M) \otimes L^2(M)$ be the unique derivation which satisfies $\delta_i(x) = x \otimes 1 - 1 \otimes x$, $\forall x \in M_i$ and $\delta_i(y) = 0$, $\forall y \in M_j$ where $j \neq i$. Then it is easy to check that δ_i defines a closable real derivation and a simple calculation (see for example Corollary 4.2 and the following remark in [Pe]) shows that the associated semigroups of c.p. maps are given by $\phi_s^1 = (e^{-2s}\mathrm{id} + (1 - e^{-2s})\tau) * \mathrm{id}$, and $\phi_s^2 = \mathrm{id} * (e^{-2s}\mathrm{id} + (1 - e^{-2s})\tau)$. In particular we have that $\{\phi_s^j\}_s$ does not converge uniformly on $(M)_1$ as $s \to 0$.

3. Approximation properties

Throughout this section δ will be a real closable derivation on a finite von Neumann algebra (N, τ), $\Delta = \delta^* \overline{\delta}$ the corresponding generator of a completely Dirichlet form, and also $\{\eta_\alpha\}_\alpha$, and $\{\phi_t\}_t$ will be the deformations described above.

In the following sections it will be necessary to consider various norms on a finite von Neumann algebra. Specifically the uniform norm, L^1-norm and L^2-norm will all play a role. We will also be considering norms on various Hilbert spaces as well. In an effort to eliminate confusion we will use the notation $\| \cdot \|_1$ to denote the L^1-norm on a von Neumann algebra, $\| \cdot \|$ will denote the uniform norm on a von Neumann algebra, while $\| \cdot \|_2$ will denote both the L^2-norm on a von Neumann algebra as well as the Hilbert space norm on a Hilbert space. The reason we have chosen this convention is because the L^2-norm on a finite von Neumann algebra (N, τ) is a pre-Hilbert space norm, and the completion of N with respect to the L^2-norm gives rise to the Hilbert N–N bimodule $L^2(N, \tau)$. As we will focus mostly on the coarse bimodule $L^2(N) \overline{\otimes} L^2(N)$, the Hilbert space norm we will mostly be interested in will be the L^2-norm coming from $N \overline{\otimes} N$.

Lemma 3.1 *If $x, y, xy \in D(\Delta)$, then $\|\Delta(x)y + x\Delta(y) - \Delta(xy)\|_1 \leq 2\|\delta(x)\|_2\|\delta(y)\|_2$.*

Proof. $\forall z \in D(\delta)$ such that $\|z\| \leq 1$ we have

$$|\tau(\Delta(x)yz + x\Delta(y)z - \Delta(xy)z)|$$
$$= |\langle \delta(x), \delta(z^* y^*) \rangle + \langle \delta(y), \delta(x^* z^*) \rangle - \langle \delta(xy), \delta(z^*) \rangle|$$

$$
\begin{aligned}
&= |\langle \delta(x), \delta(z^*y^*)\rangle + \langle \delta(y), \delta(x^*z^*)\rangle - \langle x\delta(y) + \delta(x)y, \delta(z^*)\rangle| \\
&= |\langle \delta(x), z^*\delta(y^*)\rangle + \langle \delta(y), \delta(x^*)z^*\rangle| \\
&\leq \|\delta(x)\|_2\|z^*\delta(y^*)\|_2 + \|\delta(y)\|_2\|\delta(x^*)z^*\|_2 \leq 2\|\delta(x)\|_2\|\delta(y)\|_2.
\end{aligned}
$$

As $D(\delta)$ is weakly dense the result follows by applying Kaplansky's theorem.

Lemma 3.2 *Let $\{\eta_\alpha\}_\alpha$ be the deformation described above, $\forall\alpha > 0$, $\eta_\alpha^{1/2} = \pi^{-1}\int_0^\infty \frac{t^{-1/2}}{1+t}\eta_{\alpha(1+t)/t}dt$, also $(\mathrm{id} - \eta_\alpha)^{1/2} = \pi^{-1}\int_0^\infty \frac{t^{-1/2}}{1+t}(\mathrm{id} - \eta_{t\alpha/(1+t)})dt$.*

Proof. $\forall\alpha > 0, t > 0$ we have:

$$
\begin{aligned}
\eta_\alpha(t+\eta_\alpha)^{-1} &= \eta_\alpha((t(\alpha+\Delta)+\alpha)(\alpha+\Delta)^{-1})^{-1} \\
&= \frac{1}{t}\eta_\alpha(\alpha+\Delta)\left(\frac{\alpha(1+t)}{t}+\Delta\right)^{-1} = \frac{\alpha}{t}\left(\frac{\alpha(1+t)}{t}+\Delta\right)^{-1} \\
&= \frac{1}{(1+t)}\eta_{\alpha(1+t)/t}.
\end{aligned}
$$

Hence $\eta_\alpha^{1/2} = \pi^{-1}\int_0^\infty t^{-1/2}\eta_\alpha(t+\eta_\alpha)^{-1}dt = \pi^{-1}\int_0^\infty \frac{t^{-1/2}}{1+t}\eta_{\alpha(1+t)/t}dt$. The formula for $(\mathrm{id} - \eta_\alpha)^{1/2}$ is shown similarly.

Since the range of $\eta_\alpha^{1/2}$ is the same as the domain of δ we may take the composition $\delta \circ \eta_\alpha^{1/2}$ to obtain a bounded operator from $L^2(N, \tau)$ to $\mathcal{H}$ whose norm is no more than $(2\alpha)^{1/2}$. In fact $\alpha\|x - \eta_\alpha(x)\|_2^2 \leq \|\delta \circ \eta_\alpha^{1/2}(x)\|_2^2 = \alpha\tau((x - \eta_\alpha(x))x^*) \leq \alpha\|x - \eta_\alpha(x)\|_2$, $\forall x \in N$. It will be convenient therefore to use the following notation, we will let $\zeta_\alpha = \eta_\alpha^{1/2}$, and we will let $\tilde{\delta}_\alpha = \alpha^{-1/2}\delta \circ \zeta_\alpha$. The next lemma shows that $\tilde{\delta}_\alpha$ is almost a derivation and will be a key lemma used throughout this paper. We note that in the following estimate the term $\tilde{\delta}_\alpha(a)$ will be uniformly small for $a \in F$ and thus we may omit this term from the inequality. We have chosen to keep this term however in order to emphasize the fact that $\tilde{\delta}_\alpha$ is almost a derivation.

Lemma 3.3 *Using the same notation as above if $F \subset (N)_1$, such that $\{\eta_\alpha\}_\alpha$ converges uniformly on F (F possibly infinite), then $\forall\varepsilon > 0$, $\exists\alpha_0 > 0$, such that $\forall\alpha \geq \alpha_0$ we have that $\|\tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x)\|_2^2 < \varepsilon$, and $\|\tilde{\delta}_\alpha(xa) - \tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \zeta_\alpha(x)\tilde{\delta}_\alpha(a)\|_2^2 < \varepsilon$, $\forall a \in F$, $x \in (N)_1$.*

Proof. We will prove the lemma in two parts. First we show that the vectors $\tilde{\delta}_\alpha(ax)$ and $\zeta_\alpha(a)\tilde{\delta}_\alpha(x) + \tilde{\delta}_\alpha(a)\zeta_\alpha(x)$ have approximately the same size, and then we show that the vectors have large inner product. The main difficulty is that we may not apply the product rule to a vector of the form $\alpha^{-1/2}\delta \circ \zeta_\alpha(ax)$ and thus in order to estimate the size on an inner product we must translate the expression to terms involving $\Delta^{\frac{1}{2}}$ and then use Lemmas 3.1 and 3.2 to estimate these expressions. Note that by [S3], $\Delta^{\frac{1}{2}}$ again generates a completely positive semigroup and hence is of the form $\delta_0^*\bar{\delta}_0$ for some closable derivation δ_0, thus we may apply Lemma 3.1. However some care is involved here as Lemma 3.1 only gives an estimate in $\|\cdot\|_1$ and thus we must make sure that when we apply Lemma 3.1 the term we are taking the inner product with is bounded in uniform norm.

Thus each of the parts above separates into three steps. The first step we use the properties of the derivation to set up the $\|\cdot\|_1$ estimate from Lemma 3.1, the second step we translate to terms involving $\Delta^{\frac{1}{2}}$ and use Lemma 3.1, and the third step we use Lemma 3.2 and then translate back into terms with the derivation to finish the estimate.

Let $F \subset (N)_1$ be given as above and let $\varepsilon > 0$. It follows from Lemma 3.2 and Sect. 1.1.2 of [P4] that $\exists \alpha_0 > 0$ such that $\forall \alpha \geq \alpha_0$ we have $\|a - \eta_\alpha(a)\|_2 < (\varepsilon/64)^4$, $\|a - \zeta_\alpha(a)\|_2 < \varepsilon/100$, and $\|a(\mathrm{id} - \eta_\alpha)^{1/2}(x) - (\mathrm{id} - \eta_\alpha)^{1/2}(ax)\|_2 \leq \pi^{-1}\int_0^\infty \frac{t^{1/2}}{1+t}\|a\eta_{t\alpha/(1+t)}(x) - \eta_{t\alpha/(1+t)}(ax)\|_2 < \varepsilon/100$, $\forall a \in F$, $x \in (N)_1$. Then by using the product rule for the derivation we have

$$\begin{aligned}\Big|\alpha^{-1}\|\delta(\zeta_\alpha(a)\zeta_\alpha(x))\|_2^2 - \alpha^{-1}\langle\delta(\zeta_\alpha(x)), \delta(\zeta_\alpha(a^*)\zeta_\alpha(a)\zeta_\alpha(x))\rangle\Big| \\ \leq 8\|\tilde{\delta}_\alpha(a)\|_2 \leq 8\|a - \eta_\alpha(a)\|_2^{1/2} < \varepsilon/8. \qquad (1)\end{aligned}$$

By Lemma 3.1 we have

$$\begin{aligned}&\Big|\alpha^{-1}\big\langle\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x), \Delta^{\frac{1}{2}}(\zeta_\alpha(a^*)\zeta_\alpha(a)\zeta_\alpha(x))\big\rangle \\ &\qquad - \alpha^{-1}\big\langle\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x), \zeta_\alpha(a^*)\zeta_\alpha(a)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x)\big\rangle\Big| \\ &\leq 2\alpha^{-1/2}\big\|\Delta^{\frac{1}{2}}(\zeta_\alpha(a^*)\zeta_\alpha(a)\zeta_\alpha(x)) - \zeta_\alpha(a^*)\zeta_\alpha(a)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x)\big\|_1 \\ &\leq 2\alpha^{-1/2}\big\|\Delta^{\frac{1}{2}}(\zeta_\alpha(a^*)\zeta_\alpha(a))\big\|_1 + 4\alpha^{-1/4}\big\|\Delta^{\frac{1}{4}}(\zeta_\alpha(a^*)\zeta_\alpha(a))\big\|_2 \\ &\leq 4\|a - \eta_\alpha(a)\|_2^{1/2} + 8\|a - \eta_\alpha(a)\|_2^{1/4} < \varepsilon/4. \qquad (2)\end{aligned}$$

Also from the assumptions above we have

$$\begin{aligned}&\alpha^{-1}\Big|\big\|\zeta_\alpha(a)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x)\big\|_2^2 - \big\|\Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\big\|_2^2\Big| \\ &\leq 4\alpha^{-1/2}\big\|\zeta_\alpha(a)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x) - \Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\big\|_2 \\ &\leq 8\|\zeta_\alpha(a) - a\|_2 + 4\big\|a(\mathrm{id} - \eta_\alpha)^{1/2}(x) - (\mathrm{id} - \eta_\alpha)^{1/2}(ax)\big\|_2 < \varepsilon/8. \qquad (3)\end{aligned}$$

Hence by combining (1), (2), and (3) we have shown

$$\Big|\big\|\alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x))\big\|_2^2 - \|\tilde{\delta}_\alpha(ax)\|_2^2\Big| < \varepsilon/2. \qquad (4)$$

Similarly by using the product rule we obtain that

$$\begin{aligned}\Big|\alpha^{-1}\langle\delta(\zeta_\alpha(a)\zeta_\alpha(x)), \delta(\zeta_\alpha(ax))\rangle - \alpha^{-1}\langle\delta(\zeta_\alpha(x)), \delta(\zeta_\alpha(a^*)\zeta_\alpha(ax))\rangle\Big| \\ \leq 4\|\tilde{\delta}_\alpha(a)\|_2 \leq 4\|a - \eta_\alpha(a)\|_2^{1/2} < \varepsilon/16. \qquad (5)\end{aligned}$$

Again by Lemma 3.1 we have

$$\begin{aligned}
&\left|\alpha^{-1}\left\langle \Delta^{\frac{1}{2}} \circ \zeta_\alpha(x), \Delta^{\frac{1}{2}}(\zeta_\alpha(a^*)\zeta_\alpha(ax))\right\rangle\right. \\
&\qquad\qquad\qquad \left. - \alpha^{-1}\left\langle \Delta^{\frac{1}{2}} \circ \zeta_\alpha(x), \zeta_\alpha(a^*)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\right\rangle\right| \\
&\qquad \leq 2\alpha^{-1/2}\left\| \Delta^{\frac{1}{2}}(\zeta_\alpha(a^*)\zeta_\alpha(ax)) - \zeta_\alpha(a^*)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\right\|_1 \\
&\qquad \leq 2\alpha^{-1/2}\left\| \Delta^{\frac{1}{2}} \circ \zeta_\alpha(a^*)\zeta_\alpha(ax)\right\|_1 + 4\alpha^{-1/4}\left\| \Delta^{\frac{1}{4}} \circ \zeta_\alpha(a^*)\right\|_2 \\
&\qquad \leq 2\|a - \eta_\alpha(a)\|_2^{1/2} + 4\|a - \eta_\alpha(a)\|_2^{1/4} < \varepsilon/8. \qquad (6)
\end{aligned}$$

Also from the assumptions above we have

$$\begin{aligned}
&\left|\alpha^{-1}\left\langle \zeta_\alpha(a)\Delta^{\frac{1}{2}} \circ \zeta_\alpha(x), \Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\right\rangle - \alpha^{-1}\left\| \Delta^{\frac{1}{2}} \circ \zeta_\alpha(ax)\right\|_2^2\right| \\
&\quad \leq 4\|\zeta_\alpha(a) - a\|_2 + 2\left\|a(\mathrm{id} - \eta_\alpha)^{1/2}(x) - (\mathrm{id} - \eta_\alpha)^{1/2}(ax)\right\|_2 < \varepsilon/16. \qquad (7)
\end{aligned}$$

Thus using (5), (6), and (7) we have

$$\left|\left\langle \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)), \tilde{\delta}_\alpha(ax)\right\rangle - \|\tilde{\delta}_\alpha(ax)\|_2^2\right| < \varepsilon/4. \qquad (8)$$

Hence by (4) and (8) we have that

$$\begin{aligned}
&\|\tilde{\delta}_\alpha(ax) - \zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(a)\zeta_\alpha(x)\|_2^2 \\
&= \left\|\tilde{\delta}_\alpha(ax) - \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x))\right\|_2^2 \\
&= \|\tilde{\delta}_\alpha(ax)\|_2^2 - 2\Re\left\langle \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)), \tilde{\delta}_\alpha(ax)\right\rangle + \left\|\alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x))\right\|_2^2 \\
&\leq \left|\|\tilde{\delta}_\alpha(ax)\|_2^2 - \left\|\alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x))\right\|_2^2\right| \\
&\quad + 2\left|\left\langle \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)), \tilde{\delta}_\alpha(ax)\right\rangle - \|\tilde{\delta}_\alpha(ax)\|_2^2\right| < \varepsilon.
\end{aligned}$$

The estimate for $\|\tilde{\delta}_\alpha(xa) - \tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \zeta_\alpha(x)\tilde{\delta}_\alpha(a)\|_2^2 < \varepsilon$ follows by applying the $*$-operation and using the fact that δ is a real derivation.

4. L^2-rigidity

Definition 4.1 *Let N be a finite von Neumann algebra with trace τ, if M is a finite von Neumann algebra with trace τ' such that $N \subset M$, $\tau'|_N = \tau$, and δ is a densely defined real closable derivation on M into $(L^2(M, \tau') \overline{\otimes} L^2(M, \tau'))^{\oplus\infty}$ then we say that the associated deformation $\{\eta_\alpha\}_\alpha$ is an L^2-deformation for N.*

If $B \subset N$ is a von Neumann subalgebra, the inclusion $(B \subset N)$ is L^2-rigid (or B is an L^2-rigid subalgebra of N) if any L^2-deformation for N converges uniformly on $(B)_1$. We will say that N is L^2-rigid if the inclusion $(N \subset N)$ is L^2-rigid.

Remark 4.2 1. It follows trivially that if $(B \subset N)$ is a rigid inclusion in the sense of [P4] then $(B \subset N)$ is L^2-rigid, in particular L^2-rigidity is weaker then property (T).

2. By the definition it follows that if M is a finite von Neumann algebra with normal faithful trace τ and $B \subset N \subset M$ are von Neumann subalgebras, then $(B \subset M)$ is L^2-rigid if $(B \subset N)$ is L^2-rigid.
3. If Γ is a discrete group such that $H^1(\Gamma, \ell^2\Gamma) \neq \{0\}$ then from Example 2.2 in Sect. 2 we have that $L\Gamma$ is not L^2-rigid. Also if (M_1, τ_1) and (M_2, τ_2) are finite diffuse von Neumann algebras then from Example 2.3 in Sect. 2 we have that $(M_1 * M_2, \tau_1 * \tau_2)$ is not L^2-rigid.
4. If Γ is a countable discrete group which has a proper cocycle $b : \Gamma \to (\ell^2\Gamma)^{\oplus\infty}$ (for instance $\Gamma = \mathbb{F}_n, 1 \leq n \leq \infty$) then $L\Gamma$ has no diffuse L^2-rigid von Neumann subalgebra. Indeed if $\{\eta_\alpha\}_\alpha$ is the associated deformation then $\eta_\alpha \in \mathcal{K}(L^2(L\Gamma))$, $\forall\alpha > 0$ and thus if $B \subset L\Gamma$ is a von Neumann subalgebra such that $\forall\varepsilon > 0$, $\exists\alpha_0 > 0$ such that $\forall\alpha > \alpha_0$, $x \in B_1$ we have $\|x - \eta_\alpha(x)\|_2 < \varepsilon$ then we must have that B is completely atomic (see for example Theorem 5.4 in [P4]).

Suppose $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 is infinite and Γ_2 is nonamenable, let us now sketch a simple proof that $H^1(\Gamma, \ell^2\Gamma) = \{0\}$ (see also Corollary 10 in [BV]). Suppose $b : \Gamma \to \ell^2\Gamma$ is a 1-cocycle, as Γ_2 is nonamenable $\ell^2\Gamma$ does not weakly contain the trivial representation for Γ_2 (see [MV]), hence $\exists K > 0$, $\gamma_1, \dots, \gamma_n \in \Gamma_2$ such that $\forall\xi \in \ell^2\Gamma$, $\|\xi\|_2 \leq K\sum_{i=1}^n \|\lambda(\gamma_i)\xi - \xi\|_2$. In particular we have that $\forall\gamma \in \Gamma_1$, $\|b(\gamma)\|_2 \leq K\sum_{i=1}^n \|\lambda(\gamma_i)b(\gamma) - b(\gamma)\|_2 = K\sum_{i=1}^n \|\lambda(\gamma)b(\gamma_i) - b(\gamma_i)\|_2 \leq 2K\sum_{i=1}^n \|b(\gamma_i)\|_2$. Thus we have shown that $b_{|\Gamma_1}$ is bounded and hence we may subtract from b an inner cocycle and assume that $b_{|\Gamma_1} = 0$. Therefore we have that $\forall\gamma \in \Gamma_2$, $b(\gamma)$ is a Γ_1-invariant vector, and since Γ_1 is infinite we must then have that $b(\gamma) = 0$. Thus we have shown that $b = 0$.

In Theorems 4.3 and 4.5 we will use the same idea as above to show that if $N = Q \,\overline{\otimes}\, B$ is a II_1 factor where Q is nonamenable and B is diffuse (has no minimal projections) then N must be L^2-rigid. Note that given a closable derivation δ on N there is no reason to expect that Q or B is contained in the domain of δ, thus it is necessary to use $\tilde{\delta}_\alpha$ which is everywhere defined and by Lemma 3.3 is almost a derivation. To obtain the final result we will then apply Connes' characterization of amenability [C1] which states that a factor is amenable if $\| \sum_{i=1}^n u_i \otimes_{\min} (u_i^*)^{\mathrm{op}}\| = n$, for all unitaries $u_1, \dots, u_n$ (we will use Lemma 2.2 in [H2] for the non-factor case).

Given a free ultrafilter ω, and a unital, tracial, c.p. map ϕ on a finite von Neumann algebra (N, τ) we may extend ϕ to a unital, tracial, c.p. map on N^ω by setting $\phi(x) = (\phi(x_n))_n$ if $x = (x_n)_n$. If $\{\phi_\iota\}_\iota$ is a deformation on N which does not converge uniformly on $(N)_1$ then the extension to N^ω does not converge pointwise in $\|\cdot\|_2$ to id. We will show however in the next theorem that if Q is a nonamenable subfactor then not only does an L^2-deformation converge pointwise but it actually converges uniformly to id on $(Q' \cap N^\omega)_1$.

Theorem 4.3 *Suppose (N, τ) is a finite von Neumann algebra with normal faithful trace τ and $\{\eta_\alpha\}_\alpha$ is an L^2-deformation for N. If $Q \subset N$ is a subalgebra with no non-zero amenable summands and ω is a free ultrafilter then $\{\eta_\alpha\}_\alpha$ converges uniformly on $(Q' \cap N^\omega)_1$ as $\alpha \to \infty$. In particular the inclusion $(Q' \cap N \subset N)$ is L^2-rigid.*

Proof. Take $(N, \tau) \subset (M, \tau')$ and let $\eta_\alpha : M \to M$ be an L^2-deformation. By a simple maximality argument there exists $q \in \mathcal{P}(\mathcal{Z}(Q))$ the maximal projection for which the deformation η_α converges uniformly on $(Bq)_1$ where $B = Q' \cap N^\omega$. We will show that $Q(1-q)$ is amenable.

By Lemma 2.2 in [H2] to show that $Q(1-q)$ is amenable it is enough to show that $\forall p \in \mathcal{P}(\mathcal{Z}(Q))$, $0 < p \leq 1-q$, we have

$$n = \left\| \sum_{i=1}^{n} u_i \otimes_{\min} \left(u_i^*\right)^{\mathrm{op}} \right\|, \quad \forall u_1, \ldots, u_n \in \mathcal{U}(Qp).$$

Note that as a Qp–Qp bimodule $(L^2(M) \mathbin{\overline{\otimes}} L^2(M))^{\oplus\infty}$ is just a direct sum of coarse correspondences and so the representations of Qp and Qp^{op} on $\mathcal{H}$ given by the left and right module structures induce the minimal tensor norm.

Let $p \in \mathcal{P}(\mathcal{Z}(Q))$, $0 < p \leq 1-q$, by the maximality of q we have that η_α does not converge uniformly on $(Bp)_1$, hence $\exists c > 0$ such that $\forall \alpha > 0$, $\exists x_\alpha \in (Bp)_1$ such that $\|x_\alpha - \eta_\alpha(x_\alpha)\|_2 > c$.

Let $\varepsilon > 0$, $u_1, \ldots, u_n \in \mathcal{U}(Qp)$. By Lemma 3.3 let $\alpha_0 > 0$ such that $\forall \alpha \geq \alpha_0$, $1 \leq i \leq n$, $x \in (N)_1$ we have $\|\zeta_\alpha(u_i)\tilde{\delta}_\alpha(x)\zeta_\alpha(u_i^*) - \tilde{\delta}_\alpha(u_i x u_i^*)\|_2 < c\varepsilon/2n$.

Let $x_\alpha = (x_\alpha^k)_k$ where $\|x_\alpha^k\| \leq 1$, $\forall k \in \mathbb{N}$ then $\exists k = k(\alpha) \in \mathbb{N}$ such that $\|u_i x_\alpha^k u_i^* - x_\alpha^k\|_2 < c\varepsilon/4n$, $\forall 1 \leq i \leq n$, and $\|\tilde{\delta}_\alpha(x_\alpha^k)\|_2 \geq \|x_\alpha^k - \eta_\alpha(x_\alpha^k)\|_2 \geq c$.

Therefore since $\|\tilde{\delta}_\alpha(u_i x_\alpha^k u_i^* - x)\|_2 \leq 2\|u_i x_\alpha^k u_i^* - x\|_2$ we have that

$$\begin{aligned}
n &\leq \left\|\tilde{\delta}_\alpha\left(x_\alpha^k\right)\right\|_2^{-1} \left\| \sum_{i=1}^{n} \tilde{\delta}_\alpha\left(u_i x_\alpha^k u_i^*\right) \right\|_2 + \varepsilon/2 \\
&\leq \left\|\tilde{\delta}_\alpha\left(x_\alpha^k\right)\right\|_2^{-1} \left\| \sum_{i=1}^{n} \zeta_\alpha(u_i)\tilde{\delta}_\alpha\left(x_\alpha^k\right)\zeta_\alpha(u_i^*) \right\|_2 + \varepsilon \\
&\leq \left\| \sum_{i=1}^{n} \zeta_\alpha(u_i) \otimes_{\min} (\zeta_\alpha(u_i^*))^{\mathrm{op}} \right\| + \varepsilon \\
&\leq \left\| \sum_{i=1}^{n} u_i \otimes_{\min} (u_i^*)^{\mathrm{op}} \right\| + \varepsilon.
\end{aligned}$$

Since ε was arbitrary we have that $n \leq \|\sum_{i=1}^{n} u_i \otimes_{\min} (u_i^*)^{\mathrm{op}}\|$ and thus $Q(1-q)$ is amenable.

Corollary 4.4 *Let Γ be a countable group and suppose there exists a proper cocycle $b : \Gamma \to \ell^2(\Gamma)^{\oplus\infty}$, then $L(\Gamma)$ is solid, i.e. if $B \subset L(\Gamma)$ is dif-*

fuse then $B' \cap L(\Gamma)$ is amenable. In particular all nonamenable subfactors of $L(\Gamma)$ are prime.

Proof. This follows directly from Theorem 4.3 and Remark 4.2.

We will now show that L^2-rigidity passes to normalizers of diffuse subalgebras.

Theorem 4.5 *Suppose (N, τ) is a finite von Neumann algebra with normal faithful trace τ and $\{\eta_\alpha\}_\alpha$ is an L^2-deformation for N. If ω is a free ultrafilter and $B \subset N^\omega$ is a diffuse von Neumann subalgebra such that $\{\eta_\alpha\}_\alpha$ converges uniformly on $(B)_1$, then $\{\eta_\alpha\}_\alpha$ converges uniformly on $W^*(N \cap \mathcal{N}_{N^\omega}(B))_1$. In particular if $B \subset N$ is a diffuse von Neumann subalgebra and $(B \subset N)$ is L^2-rigid, then $(W^*(\mathcal{N}_N(B)) \subset N)$ is also L^2-rigid.*

Proof. Let $1 \geq \varepsilon > 0$, using Lemma 3.3 $\exists \alpha_0 > 0$ such that $\forall \alpha > \alpha_0$, $x = (x_n)_n \in B_1$ (with $\|x_n\| \leq 1$), $y \in N_1$ we have $\lim_{n\to\omega} \|\eta_\alpha(x_n) - x_n\|_2 < \varepsilon/4$, and $\lim_{n\to\omega} \|\zeta_\alpha(x_n)\tilde{\delta}_\alpha(y) + \tilde{\delta}_\alpha(x_n)\zeta_\alpha(y) - \tilde{\delta}_\alpha(x_n y)\|_2 < \varepsilon/4$. Take $v \in N \cap \mathcal{N}_{N^\omega}(B)$ and $\alpha > \alpha_0'$, then since B is diffuse, by the mixing property of the coarse correspondence we have that $\exists u = (u_n)_n \in \mathcal{U}(B)$ (with $u_n \in \mathcal{U}(N)$) such that $\|\tilde{\delta}_\alpha(v)\|_2 \leq \lim_{n\to\omega} \|\zeta_\alpha(u_n)\tilde{\delta}_\alpha(v)\zeta_\alpha(v^* u_n^* v) - \tilde{\delta}_\alpha(v)\|_2$. Hence we have:

$$
\begin{aligned}
&\|v - \eta_\alpha(v)\|_2^2 \\
&\leq \|\tilde{\delta}_\alpha(v)\|_2^2 \\
&\leq \lim_{n\to\omega} \|\zeta_\alpha(u_n)\tilde{\delta}_\alpha(v)\zeta_\alpha(v^* u_n^* v) - \tilde{\delta}_\alpha(v)\|_2^2 \\
&\leq \lim_{n\to\omega} (\|\tilde{\delta}_\alpha(u_n)\|_2 + \|\tilde{\delta}_\alpha(u_n)\zeta_\alpha(v) + \zeta_\alpha(u_n)\tilde{\delta}_\alpha(v) - \tilde{\delta}_\alpha(u_n v)\|_2 \\
&\quad + \|\tilde{\delta}_\alpha(v^* u_n^* v)\|_2 + \|\tilde{\delta}_\alpha(u_n v)\zeta_\alpha(v^* u_n^* v) + \zeta_\alpha(u_n v)\tilde{\delta}_\alpha(v^* u_n^* v) - \tilde{\delta}_\alpha(v)\|_2)^2 \\
&< \varepsilon^2,
\end{aligned}
$$

as the maps η_α are tracial the result then follows by using the equivalence between c.p. maps and Hilbert bimodules and a standard convexity argument (see e.g. the proof of Theorem 4.2 in [P2], or the proof of Proposition 5.1 in [P4]).

Corollary 4.6 *If N is a nonamenable II_1 factor which is non-prime or has property Γ, then N is L^2-rigid.*

Proof. If $N = Q \overline{\otimes} B$ with Q a nonamenable factor then by Theorem 4.3 we have that $(B \subset N)$ is L^2-rigid. If B is diffuse then by Theorem 4.5 we then have that N is L^2-rigid.

Also if N is a nonamenable factor then by Theorem 4.3 if ω is a free ultrafilter then any L^2-deformation converges uniformly on $(N' \cap N^\omega)_1$, if N has property Γ then $N' \cap N^\omega$ is diffuse and so from Theorem 4.5 we would have that the L^2-deformation converges uniformly on $(N)_1$.

Corollary 4.7 *Let N be a finite von Neumann algebra such that N is a free product of diffuse finite von Neumann algebras or let $N = L\Gamma$ where Γ is a countable group with $H^1(\Gamma, \ell^2(\Gamma)) \neq \{0\}$.*

1. *If $B \subset N$ is a regular diffuse subalgebra then $B' \cap N$ has a non-zero amenable summand.*
2. *Any nonamenable regular subfactor of N is prime and does not have properties Γ or (T).*

Proof. 1. If $B' \cap N$ has no non-zero amenable summand then by Theorem 4.3 we would have that $(B \subset N)$ is L^2-rigid, hence by Theorem 4.5 we would have that N is L^2-rigid and thus the result follows from Remark 4.2.

2. By Corollary 4.6 and Theorem 4.5 if N has a regular subfactor which is non-prime or has properties Γ or (T) then N is L^2-rigid and so as above the result follows from Remark 4.2.

Note that if Γ is finitely generated and nonamenable then by [BV] $H^1(\Gamma, \ell^2(\Gamma)) \neq \{0\}$ if and only if $\beta_1^{(2)}(\Gamma) > 0$. For nonamenable groups which are not finitely generated it follows from a result of Gaboriau that if $H^1(\Gamma, \ell^2(\Gamma)) \neq \{0\}$ then $\beta_1^{(2)}(\Gamma) > 0$, however the reverse implication is open [MV].

5. L^2-rigid subalgebras in free product factors

Let (M_i, τ_i), $i = 1, 2$ be finite von Neumann algebras, denote $M = M_1 * M_2$. Let $\delta_i : M_1 *_{\mathrm{Alg}} M_2 \to L^2(M) \otimes L^2(M)$ be the unique derivation which satisfies $\delta_i(x) = x \otimes 1 - 1 \otimes x$, $\forall x \in M_i$ and $\delta_i(y) = 0$, $\forall y \in M_j$ where $j \neq i$. Then as above we have that $\phi_s^1 = (e^{-2s}\mathrm{id} + (1 - e^{-2s})\tau) * \mathrm{id}$, and $\phi_s^2 = \mathrm{id} * (e^{-2s}\mathrm{id} + (1 - e^{-2s})\tau)$ are the associated semigroups of c.p. maps.

If Q is an L^2-rigid subalgebra of M then we may interpret the fact that the above deformations converge uniformly on $(Q)_1$ as saying that Q has "bounded word length". Thus one would expect that a "corner of Q embeds into either M_1 or M_2" (see [P5, Theorem 2.1]). We will show in this section that this is indeed the case, we do this by first showing that Q must be rigid with respect to the deformations used in [IPP], then we may apply the word reduction argument in [IPP] (Theorem 4.3) which gives the result.

Recall that if we let $\mathcal{H}_i^0 = L^2(M_i) \ominus \mathbb{C}$ then we may decompose $L^2(M_1 * M_2)$ in the usual way as

$$L^2(M_1 * M_2) = \mathbb{C} \oplus \bigoplus_{n \geq 1} \bigoplus_{\substack{i_j \in \{1,2\} \\ i_1 \neq i_2, \ldots, i_{n-1} \neq i_n}} \mathcal{H}_{i_1}^0 \otimes \mathcal{H}_{i_2}^0 \otimes \cdots \otimes \mathcal{H}_{i_n}^0.$$

Lemma 5.1 *Let (M_1, τ_1), (M_2, τ_2) be finite von Neumann algebras. As in Sect. 2.2 of* [IPP] *denote $M = M_1 * M_2$, $\tilde{M}_j = M_j * L(\mathbb{Z})$, $j = 1, 2$, and $\tilde{M} = \tilde{M}_1 * \tilde{M}_2 = M * L(\mathbb{F}_2)$. Let $h_j \in L(\mathbb{F}_2)$ be self-adjoint elements such that $u_j = \exp(\pi i h_j)$, where $u_1, u_2 \in L(\mathbb{F}_2)$ are the canonical generators of $L(\mathbb{F}_2)$. Let $u_j^t = \exp(\pi i t h_j)$, and set $\theta_t = \mathrm{Ad}(u_1^t) * \mathrm{Ad}(u_2^t) \in \mathrm{Aut}(\tilde{M})$, a one parameter group of automorphisms. Suppose $Q \subset M$ is a von Neumann subalgebra, then the deformation $\{\theta_t\}_t$ converges uniformly on $(Q)_1$ as $t \to 0$ if and only if the deformations $\{\phi_s^j\}_s$ converge uniformly on $(Q)_1$ as $s \to 0$, $j = 1, 2$.*

Proof. Let $\varepsilon_0 > 0$ such that $\tau(u_j^t) \neq 0$, $\forall t < \varepsilon_0$, $j = 1, 2$. Let $t < \varepsilon_0$, it is then a simple exercise to check that if $f_j(t) = -\log(|\tau(u_j^t)|)$ then $\tau(\theta_t(x)x^*) = \tau(\phi^j_{f_j(t)}(x)x^*)$, $\forall x \in M_j$. In fact using the direct sum decomposition above one sees that if $x = x_1 x_2 \cdots x_n$, where $i_j \in \{1, 2\}$, $j \leq n$, $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$, and $x_j \in \mathcal{H}^0_{i_j}$, $\forall j \leq n$. Then in fact we have that

$$\begin{aligned}\tau(\theta_t(x)x^*) &= \tau\big(\theta_t(x_1)x_1^*\big) \cdots \tau\big(\theta_t(x_n)x_n^*\big)\\ &= \tau\big(\phi^{i_1}_{f_{i_1}(t)}(x_1)x_1^*\big) \cdots \tau\big(\phi^{i_n}_{f_{i_n}(t)}(x_n)x_n^*\big) = \tau\big(\phi^1_{f_1(t)} \circ \phi^2_{f_2(t)}(x)x^*\big).\end{aligned}$$

Moreover since both of the maps $\theta_{t|M}$ and $\phi^1_{f_1(t)} \circ \phi^2_{f_2(t)}$ take orthogonal vectors to orthogonal vectors we have that

$$\tau\big(\theta_t(x)x^*\big) = \tau\big(\phi^1_{f_1(t)} \circ \phi^2_{f_2(t)}(x)x^*\big), \quad \forall x \in M.$$

Since $\|\phi^1_{f_1(t)} \circ \phi^2_{f_2(t)}(x) - x\|_2 \geq \|\phi^j_{f_j(t)}(x) - x\|_2$, $\forall x \in M$, $j = 1, 2$, and since $f_j(t) \to 0$, $j = 1, 2$ as $t \to 0$ the result follows easily.

Corollary 5.2 *Let M_1 and M_2 be separable II_1 factors, and let $M = M_1 * M_2$. If $(Q \subset M)$ is L^2-rigid with Q diffuse then there exists a unique pair of projections $q_1, q_2 \in Q' \cap M$ such that $q_1 + q_2 = 1$, and $u_i(Qq_i)u_i^* \subset M_i$ for some unitaries $u_i \in \mathcal{U}(M)$, $i = 1, 2$. Moreover, these projections lie in the center of $Q' \cap M$.*

Proof. Suppose $(Q \subset M)$ is L^2-rigid, then by definition we have that the deformations $\{\phi_s^j\}_s$, converge uniformly to id on $(Q)_1$ as $s \to 0$, hence by Lemma 5.1 the deformation $\{\theta_t\}_t$ also converges uniformly on $(Q)_1$ as $t \to 0$. A check of Theorem 4.3 in [IPP] shows that these are the only two facts used from the rigid inclusion. Thus the result follows from Theorems 4.3 and 5.1 in [IPP].

6. Unique prime factorization and non-L^2-rigid factors

In this section we will adapt Theorems 4.3 and 4.5 and use Popa's intertwining technique along with the results in [OP] in order to show that if M_i are II_1 factors which have derivations into $L^2(M_i) \otimes L^2(M_i)$ which do not "vanish" then the tensor product has unique prime factorization (up to amplification and unitary conjugation of the factors). In order to satisfy the conditions of Popa's intertwining criteria (Theorem 2.1 in [P5]) it will be necessary to assume that the derivation is actually densely defined on M_i. This is a formally stronger condition then the negation of L^2-rigidity, however note that both Examples 2.2 and 2.3 in Sect. 2 satisfy this condition.

For the following theorem if $M = M_1 \overline{\otimes} M_2 \overline{\otimes} \cdots \overline{\otimes} M_m$ then we will denote by $\hat{M}_i$ the resulting von Neumann subalgebra obtained by replacing M_i with $\mathbb{C}1$ so that $M = M_i \overline{\otimes} \hat{M}_i$,

Theorem 6.1 *Let M_i be nonamenable II_1 factors $1 \leq i \leq m$, suppose that each M_i has a densely defined real closable derivation into $(L^2(M_i) \otimes L^2(M_i))^{\oplus\infty}$ such that the associated L^2-deformation does not converge uniformly on $(M_i)_1$. Let $M = M_1 \overline{\otimes} M_2 \overline{\otimes} \cdots \overline{\otimes} M_m$. Assume that $B \subset M$ is a regular type II_1 factor such that $B' \cap M$ is a nonamenable subfactor. Then $\exists k \in \{1, \ldots, m\}$, $t > 0$ and a unitary element $u \in \mathcal{U}(M)$ such that $uBu^* \subset (\hat{M}_k)^t \otimes \mathbb{C} \subset (\hat{M}_k)^t \overline{\otimes} (M_k)^{1/t} = M$. If in addition we have that the L^2-deformations above may all be taken compact then the same conclusion follows if we drop the hypothesis that B is regular.*

Proof. Let $\delta_0^i : M_i \to (L^2(M_i) \overline{\otimes} L^2(M_i))^{\oplus\infty}$ be a densely defined closable real derivation such that the corresponding deformation $\{\eta_\alpha^i\}$ does not converge uniformly on $(M_i)_1$. Then we may embed $(L^2(M_i) \overline{\otimes} L^2(M_i))^{\oplus\infty}$ into $\mathcal{H}_i = (L^2(M) \overline{\otimes}_{\hat{M}_i} L^2(M))^{\oplus\infty}$ in the natural way as M_i–M_i Hilbert bimodules and we then may extend δ_0^i to a densely defined closable real derivation δ^i on M by setting $\delta^i(x) = 0$, $\forall x \in \hat{M}_i$. We denote by $\{\hat{\eta}_\alpha^i\}$ the corresponding deformations on M, so that $\hat{\eta}_\alpha^i = \eta_\alpha^i \otimes \mathrm{id}$, also let $\hat{\zeta}_\alpha^i = \zeta_\alpha^i \otimes \mathrm{id} = (\hat{\eta}_\alpha^i)^{1/2}$.

We will proceed as in Theorem 4.3 to show that if each $\{\hat{\eta}_\alpha^i\}$ does not converge uniformly on $(B)_1$ then we must have that $Q = B' \cap M$ is amenable. Indeed if this is the case then $\exists c > 0$, such that $\forall \alpha > 0$, $i \leq m$, $\exists x_\alpha^i \in (B)_1$ such that $\|x_\alpha^i - \hat{\eta}_\alpha^i(x_\alpha^i)\|_2 \geq c$. Let $\varepsilon > 0$, $u_1, \ldots, u_n \in \mathcal{U}(Q)$. By Lemma 3.3 let $\alpha_0 > 0$ such that $\forall \alpha \geq \alpha_0$, $x \in (B)_1$, $1 \leq i \leq n$ we have $\|\hat{\zeta}_\alpha(u_i)\tilde{\delta}_\alpha^i(x)\hat{\zeta}_\alpha(u_i^*) - \tilde{\delta}_\alpha^i(u_i x u_i^*)\|_2 < c^m\varepsilon/2nm$ and $\|\check{\zeta}_\alpha^i(u_i)x\check{\zeta}_\alpha^i(u_i^*) - x\|_2 < c^m\varepsilon/4nm$, where $\hat{\zeta}_\alpha = \zeta_\alpha^1 \circ \cdots \circ \zeta_\alpha^m$ and $\check{\zeta}_\alpha^i$ is obtained by omitting ζ_α^i from $\hat{\zeta}_\alpha$.

Let ${}_M\mathcal{H}_M = \mathcal{H}_1 \overline{\otimes}_M \mathcal{H}_2 \overline{\otimes}_M \cdots \overline{\otimes}_M \mathcal{H}_m$, and note that $\mathcal{H}$ may be embedded into $(L^2(M) \overline{\otimes} L^2(M))^{\oplus\infty}$ as M–M Hilbert bimodules. Let $\xi_\alpha = \tilde{\delta}_\alpha^1(x_\alpha^1) \otimes \cdots \otimes \tilde{\delta}_\alpha^m(x_\alpha^m) \in \mathcal{H}$ then we have that $\|\xi_\alpha\|_2 \geq c^m$ and following the same proof as in Theorem 4.3 we have that $n \leq \|\sum_{i=1}^n u_i \otimes_{\min} (u_i^*)^{\mathrm{op}}\| + \varepsilon$. Since ε was arbitrary we have that Q is amenable.

Therefore if $B' \cap M$ is a nonamenable factor then we have shown that $\exists k \leq m$ such that the deformation $\{\hat{\eta}_\alpha^k\}$ converges uniformly on $(B)_1$. Next we show that if this is the case then we have that a corner of B embeds into $\hat{M}_k$ inside of M, i.e. there exists a non-zero projection f in $B' \cap \langle M, e_{\hat{M}_k} \rangle$ of finite trace $Tr = Tr_{\langle M, e_{\hat{M}_k} \rangle}$.

If we do not have that a corner of B embeds into $\hat{M}_k$ inside of M then by Corollary 2.3 of [P5] there exists a sequence of unitaries $\{u_n\}_n \subset \mathcal{U}(B)$ such that $\forall x \in M$, $\|E_{\hat{M}_k}(xu_n)\|_2 \to 0$, as $n \to \infty$. Since $\hat{\zeta}^k_{\alpha|\hat{M}_k} = \mathrm{id}$ we have that $\forall x \in M$, $\|E_{\hat{M}_k}(x\hat{\zeta}^k_\alpha(u_n))\|_2 \to 0$, as $n \to \infty$, and since $\hat{M}_k$ is regular in M this implies $\|E_{\hat{M}_k}(x\hat{\zeta}^k_\alpha(u_n)y)\|_2 \to 0$, as $n \to \infty$, $\forall x, y \in M$. In particular this shows that $\forall v \in \mathcal{N}_M(B)$, $\exists u \in \mathcal{U}(B)$ such that

$$\left\|\zeta^k_\alpha(u)\tilde{\delta}^k_\alpha(v)\zeta^k_\alpha(v^*u^*v) - \tilde{\delta}^k_\alpha(v)\right\|_2 \geq \left\|\tilde{\delta}^k_\alpha(v)\right\|_2 \tag{9}$$

On the other hand since B is regular and since $\{\eta^k_\alpha\}_\alpha$ does not converge uniformly on $(M)_1$, $\exists c_0 > 0$ such that $\forall \alpha > 0$, $\exists v_\alpha \in \mathcal{N}_M(B)$ such that $\|\tilde{\delta}^k_\alpha(v_\alpha)\|_2 \geq \|v_\alpha - \eta^k_\alpha(v_\alpha)\|_2 \geq c_0$. By Lemma 3.3 $\forall \varepsilon > 0$, $\exists \alpha_0 > 0$ such that $\forall \alpha \geq \alpha_0$, $u \in \mathcal{U}(B)$ we have that

$$\left\|\zeta^k_\alpha(u)\tilde{\delta}^k_\alpha(v_\alpha)\zeta^k_\alpha(v^*_\alpha u^* v_\alpha) - \tilde{\delta}^k_\alpha(v_\alpha)\right\|_2 < \varepsilon.$$

Thus for $\varepsilon < c_0$ we have

$$\left\|\zeta^k_\alpha(u)\tilde{\delta}^k_\alpha(v_\alpha)\zeta^k_\alpha(v^*_\alpha u^* v_\alpha) - \tilde{\delta}^k_\alpha(v_\alpha)\right\|_2 < \left\|\tilde{\delta}^k_\alpha(v_\alpha)\right\|_2,$$

for each $u \in \mathcal{U}(B)$, which contradicts (9).

If B is not regular but each deformation is compact then we may apply the proof of Theorem 6.2 in [P4] to show that a corner of B embeds into $\hat{M}_k$ inside of M in this case also.

Thus in either case since $B' \cap M$ is a factor we may then apply Proposition 12 in [OP] to obtain the result.

As a consequence of the previous theorem, we obtain from [OP] the following unique prime factorization result.

Corollary 6.2 *Let M_i be nonamenable II_1 factors $1 \leq i \leq m$, suppose that each M_i has a densely defined real closable derivation into $(L^2(M_i) \otimes L^2(M_i))^{\oplus\infty}$ such that the associated L^2-deformation does not converge uniformly on $(M_i)_1$. Assume $N_1 \overline{\otimes} \cdots \overline{\otimes} N_n = M_1 \overline{\otimes} \cdots \overline{\otimes} M_m$, for some prime II_1 factors $N_1, \ldots, N_n$, then $n = m$ and there exist $t_1, t_2, \ldots, t_m > 0$ with $t_1 t_2 \cdots t_m = 1$ such that after permutation of indices and unitary conjugacy we have $N_k^{t_k} = M_k$, $\forall k \leq m$.*

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